DISCRETE REAL TIME FLOWS WITH QUASI-DISCRETE SPECTRA AND ALGEBRAS GENERATED BY exp $q(t)$

BY

F. HAHN*

ABSTRACT

These notes (essentially unedited) were sent to W. Parry in 1964. The first two parts are complete and in a letter to Parry at that time Hahn indicated his intention to publish them. Evidently he did not manage to do this. The remainder of these notes represents an attempt to establish a theory of quasi-discrete spectra for discrete one-parameter flows. Hahn indicates the gaps and in a following note Parry clarifies his theory. The first part of these notes presents a characteristic example of a discrete one-parameter flow with quasi-discrete spectrum. Ergodicity, minimality and distality are established. The second part examines the Banach algebra of functions on R generated by $\{\exp q(t) : q \text{ a real polynomial of degree } < n+1\}$ and shows that the shift isometries arise from a discrete one-parameter flow on its maximal ideal space Λ_n and that if n is finite this flow is isomorphic to the example examined in the first part.

Introduction

We let X be a compact topological abelian group. We say that T is an affine transformation if there is an automorphism S of X and an element $x_0 \in X$ such that $T(x) = x_0 \cdot S(x)$. We ask the question as to when is it possible to find a one parameter group T_t, t a real number, of affine transformations of X given by $T_t(x) = x_t \cdot S_t(x)$. Since $T_{t+s} = T_t T_s$ we see that

(0.1) $$S_{t+s} = S_t S_s$$

(0.2) $$x_{t+s} = x_t S_t(x_s)\,.$$

These conditions are necessary and sufficient for the T_t to form a one parameter group. Since Iwasawa has shown that the automorphism group of X is totally disconnected it follows that the family T_t cannot be continuous in t. Thus to

* Deceased

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begin with it is not clear that one parameter groups of affine transformations exist. We intend to exhibit a family of examples of one parameter groups of affine transformations. These will be analogous to the single affine transformation with quasi-discrete spectrum. We will then study this notion from the abstract point of view. Finally we consider the relation of these transformations to the compactification of the reals with respect to a particular function algebra.

1. The example

We let Γ be the Bohr compactification of the reals. We let R_d be the additive group of reals with the discrete topology. Let $X = \Gamma \times \cdots \times \Gamma$ (n times); then $\hat{X} = R_d \times \cdots \times R_d$ (n times) where $\hat{X}$ is the dual group of X. If we use $\gamma = (\gamma_1 \cdots \gamma_n)$ to designate elements of X and $v = (v_1, \cdots, v_n)$ to designate the elements of $\hat{X}$ and (v, γ) to mean the value of v at γ then

$$(v, \gamma) = \sum_{i=1}^{n} (v_i, \gamma_i).$$

We define now a one parameter group, $\hat{S}_t$ of automorphisms of $\hat{X}$. If $\mu = \hat{S}_t v$ then

$$\begin{aligned}
\mu_n &= v_n \\
\mu_{n-1} &= P_1(t)v_n + v_{n-1} \\
\mu_{n-2} &= P_2(t)v_n + P_1(t)v_{n-1} + v_{n-2} \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\mu_{n-j} &= P_j(t)v_n + P_{j-1}(t)v_{n-1} + \cdots + v_{n-j} \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\mu_1 &= P_{n-1}(t)v_n + P_{n-2}(t)v_{n-2} + \cdots + v_1
\end{aligned}$$

where

$$P_k(t) = \frac{t(t-1)\cdots(t-k+1)}{k!}, \quad k = 1, 2, 3 \cdots$$

$$P_0(t) = 1.$$

We see easily that for each t the mapping $\hat{S}_t$ is an automorphism of $\hat{X}$.

We now wish to show that $S_{t+s} = S_t \cdot S_s$. This actually is a direct result of the following equation.

$$P_k(s+t) = P_0(s)P_k(t) + P_1(s)P_{k-1}(t) + \cdots + P_k(s)P_0(t) . \tag{1.1}$$

If s and t are integers the equation says that the number of ways of choosing k objects from $s + t$ objects ($P_k(s+t)$) is equal to the number of ways of choosing none from the s group and k from the t group plus the number of ways of choosing

1 from the s group and $k-1$ from the t group plus etc. Since in both sides of this equation all expressions are polynomials in s and t, and since the equation is true for all integral values of s and t it must hold true for all s and t. If we let $w = \hat{S}_s\hat{S}_t(v)$ then

$$\begin{aligned} w_{n-j} = {} & [P_j(s)P_0(t) + P_{j-1}(s)P_1(t) + \cdots + P_0(s)P_j(t)]v_n \\ & + [P_{j-1}(s)P_0(t) + P_{j-2}(s)P_1(t) + \cdots + P_0(s)P_{j-1}(t)]v_{n-1} \cdots \\ & \cdots + P_0(s)P_0(t)v_{n-j}. \end{aligned}$$

If we let $\mu = \hat{S}_{s+t}(v)$ we have

$$\mu_{n-j} = P_j(t+s)v_n + P_{j-1}(t+s)v_{n-1} + \cdots + v_{n-j}.$$

Using (1.1) and comparing the above equation we have $\hat{S}_{s+t} = \hat{S}_s\hat{S}_t$. We let S_t be the dual automorphism of $\hat{S}_t$. That is $(S_t\gamma, v) = (\gamma, \hat{S}_t v)$ for $\gamma \in X$ and $v \in \hat{X}$. We immediately see that

(1.2) $$S_{s+t} = S_s \cdot S_t.$$

Our next task is to find a family $\gamma(t) \in X$ for which (0.2) is satisfied. Let $\alpha(t) \in \Gamma$ be defined by the following: $(\alpha(t), s) = e^{ist}$ for each real number t and $s \in R_d$. This is a dense one parameter subgroup of Γ. It is immediate that $\alpha(t+s) = \alpha(t)\alpha(s)$. To see that it is dense we need only observe that if $n \in R_d$ and $(\alpha(t), n) = 1$ for all t then $n = 0$. We define $\gamma(t)$ as follows

(1.3) $$\gamma(t) = (\alpha(t), \alpha(P_2(t)), \alpha(P_3(t)) \cdots \alpha(P_n(t)).$$

We now must show that (0.2) holds, i.e., $\gamma(t+s) = \gamma(t) \cdot S_t(\gamma(s))$. This is equivalent to showing that

$$\begin{aligned} (\gamma(t+s), v) &= (\gamma(t)\, S_t(\gamma(s)), v) \\ &= (\gamma(t), v)(\gamma(s), \hat{S}_t v) \end{aligned}$$

for all $v \in \hat{X}$.

If we let $\exp x = e^{ix}$ then we have

$$(\gamma(t+s), v) = \exp\,(v_1P_1(t+s) + v_2P_2(t+s) + \cdots + v_nP_n(t+s))$$

and

$$(\gamma(t), v) = \exp\,(v_1P_1(t) + v_2P_2(t) + \cdots + v_nP_n(t))$$

and

$$(\gamma(s), \hat{S}_t v) =$$

$$\begin{aligned} &\exp\{P_1(s)[P_{n-1}(s)v_n + P_{n-2}(s)v_{n-1} + \cdots P_1(s)v_2 + v_1] \\ &+ P_2(s)[P_{n-2}(t)v_n + P_{n-3}(t)v_{n-1} + \cdots v_2] + \cdots + P_n(s)P_0(t)v_n\}. \end{aligned}$$

Comparing the three equations above and using (1.1) we see that (0.2) holds. We have thus shown that if

$$T_t(\gamma) = \gamma(t) \cdot S_t(\gamma) \text{ then} \tag{1.4}$$

$$T_{t+s} = T_t \circ T_s.$$

We wish to examine some of the properties of the dynamical system (X, T_t) which we have just described. In order to do this we establish some convention. Let $R_d^j \subset \hat{X}$ for $j = 0, 1, 2 \cdots n-1$ be the set of all $v \in \hat{X}, v = (v_1, v_2, \cdots v_j, v_{j+1} \cdots v_n)$ where $0 = v_{j+2} = v_{j+2} = \cdots = v_n$ and let $R_d^n = \hat{X}$. We define $\Gamma^j \subset X$ for $j = 0, 1, 2 \cdots n$ in a similar fashion. We let $\hat{R}_t$ be defined by the equation

$$\hat{R}_t = \hat{S}_t - I. \tag{1.5}$$

Thus we obtain

$$R_t(\gamma) = \gamma^{-1} S_t(\gamma). \tag{1.6}$$

Since the T_t form a one parameter group we see that $\hat{R}_t$ and R_t satisfy the resolvent equations

$$\hat{R}_t \cdot \hat{R}_s = \hat{R}_{t+s} - \hat{R}_t - \hat{R}_s \tag{1.7a}$$

$$R_t \cdot R_s(\gamma) = R_{t+s}(\gamma)[R_t(\gamma)]^{-1}[R_s(\gamma)]^{-1}. \tag{1.7b}$$

Again from the fact that the T_t form a one parameter group we obtain

$$\gamma(t+s) = \alpha(t) \cdot \alpha(s) \cdot R_t(\alpha(s)). \tag{1.8}$$

THEOREM 1. *Let (X, T_t) be defined as in the preceding paragraphs. Then (X, T_t) is ergodic.*

PROOF. Let H_1 be the subspace of $L^2(X)$ spanned by the functions determined by $\mathbb{R}_d^1$. These are all functions, of the form $f_v(\gamma) = (\gamma, v)$ where $v \in \mathbb{R}_d^1$. The orthogonal complement H_2 of H_1 is the subspace determined by $\mathbb{R}_d^n \backslash \mathbb{R}_d^1$ (set theoretic complement). We define the unitary operation $V_t : L^2(X) \to L^2(X)$ by $V_t f = f \circ T_t$ for all $f \in L^2(X)$ and t a real number. We observe that if f_v is determined by $v \in \mathbb{R}_d^n \backslash \mathbb{R}_d^1$ then $V_t f_v = \lambda\, g\, f_v$ where g is a non-constant character and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. We then have

$$(V_t f_v, f_v) = 0 \text{ for } v \in \mathbb{R}_d^n \backslash \mathbb{R}_d^1.$$

Using continuity and linearity arguments we obtain

$$\lim_{t \to \pm\infty} (V_t f, f) = 0 \text{ for } f \in H_2.$$

We now will show that (X, T_t) is ergodic by showing that V_t has only constants in the eigenspace one. Let $f \in L^2(X)$ and suppose $V_t f = f$, $f = f_1 + f_2$ where $f_1 \in H_1$ and $f_2 \in H_2$. Since H_1 and H_2 are stable under V_t and $V_t f = f$, we get $V_t f_1 = f_1$ and $V_t f_2 = f_2$. Since $(V_t f_2, f_2) \to 0$ we see that $f_2 = 0$. Thus $f \in H_1$ and is a function of the variable γ_1 alone. We have $V_t f(\gamma_1) = f(\alpha(t)\gamma_1)$. Since $\{\alpha(t)\}$ is a dense subgroup of Γ we see that f is constant almost everywhere and this concludes the proof.

THEOREM 2. *The system (X, T_t) is distal.*

PROOF. We suppose that $\{T_{t_m}\beta : m \in \Delta\}$ and $\{T_{t_m}\gamma : m \in \Delta\}$ are two nets in X such that there is a $z \in X$ for which $T_{t_m}\beta \to z$ and $T_{t_m}\gamma \to z$. We must show that $\beta = \gamma$ in order to prove our theorem. Since $T_{t_m}(\beta) = \gamma(t_m) S_{t_m}(\beta)$ and since X is a group we see that $S_{t_m}(\gamma\beta^{-1}) \to e$. Thus it is enough to show that if $S_{t_m}(\alpha) \to e$ then $\alpha = e$. We compute this from the definition of S_{t_m}. If $S_{t_m}(\alpha) \to e$ then for every $v \in \hat{X}$ we have $(S_{t_m}(\alpha), v) = (\alpha, \hat{S}_{t_m} v) \to 1$. If $\alpha = (\alpha_1, \alpha_2 \cdots \alpha_n)$ and $v = (v_1 \cdots v_n)$ then

$$\begin{aligned}\hat{S}_{t_m}(v) = (&v_n P_{n-1}(t_m) + \cdots + P_1(t_m) v_2 + v_1,\\ &v_n P_{n-2}(t_m) + \cdots + v_2,\\ &\cdot\\ &\cdot\\ &\cdot\\ &v_n).\end{aligned}$$

Our procedure is inductive. Let $o = v_2 = v_3 = \cdots = v_n$ then $(S_{t_m}(\alpha), v) = (\alpha_1, v_1) \to 1$ for all $v_1 \in \mathbb{R}_d$ so that $a_1 = e$. The argument proceeds step by step up to n showing that $\alpha = e$ and completing the proof.

THEOREM 3. *(X, T_t) is minimal.*

PROOF. Since (X, T_t) is distal we know from Ellis' theorem that each orbit is almost periodic (using the discrete topology on the parameter t). Thus to show minimality we need only show that there is a dense orbit. We wish to show that the points $T_t(e)$ are uniformly distributed. This is of course even stronger than density. We see this as follows: $T_t(e) = \gamma(t) = (\alpha(t), \alpha(P_2(t)), \cdots, \alpha_n(P_n(t))$. Let $v = (v_1, \cdots, v_n)$ be a character of X, then $(v, \gamma(t)) = \exp\ (v_1 t + v_2 P_2(t) + \cdots + v_n P_n(t))$. If $v \neq o$ then it is easy to see by a well-known result of Weyl that $\lim_{T-\infty}(1/2T \int_{-T}^{T} (v, \gamma(t))\, dt \to 0$.

But this is the criterion for uniform distribution and the theorem is complete.

We now indicate that the preceding construction can be carried out in the case of infinite product spaces. Let $X = \Pi_1^\infty \Gamma$ be the unrestricted product of the Γ's. Then $\hat{X} = \Sigma_1^\infty \mathbb{R}_d$ the restricted product or finite direct sum of the $\mathbb{R}_d$. Since each element $v \in \hat{X}$ has only finitely many non-zero coordinates we may write $v = (v_1, v_2, \cdots v_n, \text{o}, \text{o}, \cdots)$ and $\hat{S}_t$ is defined as before. We also define $\gamma(t) = (\alpha(t), \alpha(P_2(t)) \cdots \cdots \alpha(P_n(t) \cdots)$ and let $T_t(\gamma) = \gamma(t) S_t(\gamma)$ as before. We again see that (X, T_t) in an ergodic distal minimal dynamical system.

2. Compactifications of $\{\exp q_n(t)\}$

Let $C(\mathbb{R})$ be the Banach algebra of all bounded complex valued continuous functions on the real numbers. By Λ_n we shall mean the closed subalgebra generated by all functions of the form $\exp q(s)$ where $q(s)$ is a polynomial with real coefficients of degree less than or equal to n. We let $n = \infty$ and define Λ_∞ to be the closed subalgebra generated by all functions of the form $\exp q(t)$ where q is a polynomial with real coefficients. We define the transformation U_t on $C(\mathbb{R})$ as follows $U_t f(s) = f(s + \text{t})$. Each U_t is an isometry of $C(\mathbb{R})$ and $U_{t+n} = U_t \circ U_n$. The family is however not continuous in the parameter t since we allow non uniformly continuous functions in $C(\mathbb{R})$.For each n (Λ_n, U_t) forms a dynamical system.

For each n we let Y_n be the compactification of $\mathbb{R}$ with respect to Λ_n. That is, Y_n is characterized by the following: Y_n is compact and there is a one-to-one continuous map $\eta : R \to Y_n$ whose image is dense in Y_n. If $f \in \Lambda_n$ then $f(\eta(s))$ may be extended to a continuous function of Y_n. Also if $\tilde{f} \in C(Y_n)$ then there is an $f \in \Lambda$ for which $\tilde{f}(\eta(s)) = f(s)$. Under such circumstances each U_t induces a homeomorphism on Y_n. We again call this by the name U_t. Then (Y_n, U_t) forms a dynamical system.

We use (Γ_n, T_t) to indicate the systems described in the previous section where $\Gamma_n = \Pi_1^n \Gamma$.

THEOREM 2.1. *(Y_n, U_t) is isomorphic to (Γ_n, T_t).*

PROOF. Let $B(\Gamma_n)$ be the set of all continuous functions on the reals obtained as follows: $f \in B(\Gamma_n)$ if there is an $\tilde{f} \in C(\Gamma_n)$ such that $f(s) = \tilde{f}(T_s(e))$. It is not hard to see that in order to prove our theorem we need only show that (Λ_n, U_t) and $(B(\Gamma_n), T_t)$ are identical. From the definitions of the action of U_t and T_t we need only show that $\Lambda_n = B(\Gamma_n)$. Let $v = (v_1 \cdots, v_j, \cdots)$ be a character on Γ_n. Consider

$(v, T_s(e)) = (v, \gamma(s)) = \exp(v_1 s + v_2 P_2(s) + v_3 P_3(s) \cdots v_j P_j(s) \cdots)$. Since only finitely many v_i are not zero we see that v restricted to $T_s(e)$ is in Λ_n. Since the characters generate $C(\Gamma_n)$ we see that $B(\Gamma_n) \subset \Lambda_n$. We need only show that if $\exp q(t) \in \Lambda_n$ then there is an $\tilde{f} \in C(\Gamma_n)$ for which $\tilde{f}(T_t(e)) = \exp q(t)$. Now $q(t)$ has degree $j \leqq n$ if n is finite and $j < n$ if $n = \infty$. Choose a character $v = (v_1, \cdots, v_j, 0, \cdots)$. Then $(v, T_t(e)) = \exp(v_1 t + v_2 P_2(t) + \cdots + v_j P_j(t))$. Since the degree of $P_k(t)$ is exactly k we see that v may be chosen such that $(v, T_t(e)) = \exp q(t)$. Thus $\Lambda_n \supset B(\Gamma_n)$ showing that $\Lambda_n = B(\Gamma_n)$ and completing the proof.

3. Quasi-discrete spectrum for one parameter dynamical systems

We now assume that X is a compact Hausdorff space and that for each $t \in R_d$ we have a homeomorphism $T_t : X \to X$ such that $T_{t+s} = T_t \circ T_s$. We consider the system (X, T_t). Notice we do not assume continuity in the parameter t. We further assume that (X, T_t) is minimal. We now wish to define quasi-eigen functions and quasi-eigenvalues for (X, T_t). Let $V_t f = f \cdot T_t$ for each $f \in C(X)$

We consider first all $f \in C(X)$ for which $V_t f = \lambda(t) f$. Since V_t is an isometry of $C(X)$ it follows that $|\lambda(t)| = 1$ and thus by minimality that $|f|$ is constant. With this in mind we let G_1 be the set of all f for which $|f| = 1$ and $V_t f = \lambda(t) f$ where $\lambda(t) \in C$. We see easily that G_1 is a group under multiplication. We let H_1 be the set of all functions $\lambda : R_d \to C$ such that $|\lambda(t)| = 1$ and there is $f \in G$, for which $V_t f = \lambda(t) f$. Since $V_{t+s} = V_t \circ V_s$ we see that $\lambda(t+s) = \lambda(t) \cdot \lambda(s)$ so that λ is a character of R_d. Suppose we have already defined $H_1 \subset H_2 \subset \cdots \subset H_n$, $G_1 \subset G_2 \subset \cdots \subset G_n$ with the following properties. Each G_i is a group of functions of modulus one contained in $C(X)$. Each element f of H_i is a function $f : R_d \times X \to \mathbb{C}$, f is of modulus 1. Furthermore if t is fixed then $f(t, \,) \in G_{i-1}$ and there is a $g \in G$ for which $V_t g = f(t, \,) g$. Each $g \in G_i$ has the property that $V_t g = f(t, \,) g$ and $f \in H_i$.

We now define G_{n+1} and H_{n+1}. We let G_{n+1} be the set of all functions g of modulus one such that for each $t \in R_d$ there is an $f_t \in G_n$ such that $V_t g = f_t g$. We let H_{n+1} be the set of all functions $f : R_d \times X \to \mathbb{C}$ which have the property that there is $g \in G_{n+1}$ for which $f(t, x) = f_t(x)$ where $V_t g = f_t g$. All the previous properties are preserved. We also note that the H_i are groups under multiplication. Let $H = \bigcup_{i=1}^{\infty} H_i$ and $G = \bigcup_{i=1}^{\infty} G_i$.

We observe that it is a consequence of the minimality of (X, T_t) that if two quasi-eigen functions have identical quasi-eigenvalues then one is a constant multiple of the other. For suppose $V_t g_1 = f_t g_1$ and $V_t g_2 = f_t g_2$ then $V_t g_1 g_2^{-1} =$

$g_1 g_2^{-1}$ and thus $g_1 g_2^{-1}$ is constant. On G we define the following homomorphisms:

$$R_t g = g^{-1} V_t g. \tag{3.1}$$

That is, if $V_t g = f_t g$ then $R_t g = f_t$. For each fixed t we have $f_t \in G_{n-1}$ if $g \in G_n$. We observe

$$R_t R_s = R_s R_t, \tag{3.2}$$

for
$$\begin{aligned} R_s(R_t g) &= (R_t g)^{-1}\, V_s(R_t g) \\ &= (g^{-1} V_t g)^{-1}\, V_s(g^{-1} V_t g) \\ &= g\, V_t g^{-1} (V_s g)^{-1}\, V_{s+t} g \\ &= (g^{-1}\, V_s g)^{-1}\, V_t (g^{-1} V_s g) \\ &= (R_s g)^{-1}\, V_t(R_s g) \\ &= R_t(R_s g). \end{aligned}$$

On H we wish to define a homomorphism $\tilde{R}_t : H \to H$ such that $\tilde{R}_t H_1 = e$ and $\tilde{R}_t H_n \subset H_{n-1}$.

If $f \in H_n$ then there is a $g \in G_n$ such that $f(s,x) = (R_s g)(x)$.

$$\text{Let } \tilde{R}_t f(s,x) = (R_t(R_s g)(x). \tag{3.3}$$

This mapping is well defined. We must show that $\tilde{R}_t f \in H_{n-1}$. That is we must show that for each fixed t there is a function $h \in G_{n-1}$ for which $(R_s g)(x) = \tilde{R}_t f(s,x)$. We have $\tilde{R}_t f(s,x) = (R_t(R_s g))(x) = (R_s(R_t g))(x)$. Thus letting $h = R_t g$ we see that $\tilde{R}_t f \in H_{n-1}$.

We see that $\tilde{R}_t$ is a homomorphism as follows

$$\begin{aligned} \tilde{R}_t(f_1 f_2) = R_t(R_s(g_1 g_2)) &= R_t(R_s g_1 \cdot R_s g_2) \\ &= R_t(R_s g_1) \cdot R_t(R_s g_2) = \tilde{R}_t f_1 \cdot \tilde{R}_t f_2. \end{aligned}$$

RESOLVENT EQUATION. *We first wish to observe that the homomorphisms $R_t : G \to G$ satisfy the resolvent equation*

$$R_t(R_n(g)) = [R_t(g)]^{-1} [R_n(g)]^{-1} R_{t+n}(g). \tag{3.4}$$

We compute this as follows

$$\begin{aligned} R_t(R_n(g)) &= [R_n(g)]^{-1}\, V_t[R_n(g)] \\ &= [R_n(g)]^{-1} [V_t(g)]^{-1}\, V_{t+n}(g) \\ &= [R_n(g)]^{-1} [g^{-1} V_t(g)]^{-1}\, g^{-1} V_{t+n}(g) \\ &= [R_n(g)]^{-1}\; [R_t(g)]^{-1}\, R_{t+n}(g). \end{aligned}$$

From this we wish to conclude that the $\tilde{R}_t$ satisfy the resolvent equation

$$\tilde{R}_t(\tilde{R}_n(f)) = [\tilde{R}_t(f)]^{-1}[\tilde{R}_n(f)]^{-1}\tilde{R}_{t+n}(f). \tag{3.5}$$

To see this we let $f \in H$ and $g \in G$ such that $f(s,x) = (R_s g)(x)$. For each fixed n let $g_n = R_n g$ and for each n and s let $f_n(s,x) = (\tilde{R}_n f)(s,x)$. We have

$$\begin{aligned} f_n(s,x) &= (\tilde{R}_n f)(s,x) = (\tilde{R}_n(R_s g))(x) \\ &= (R_s(R_n g))(x) = (R_s(g_n))(x). \end{aligned}$$

The above is true for each fixed s and for each fixed n. Thus we obtain:

$$\begin{aligned} (\tilde{R}_t(\tilde{R}_n f))(s,x) &= (\tilde{R}_t(f_n))(s,x) \\ &= (R_t(R_s g_n))(x) = (R_s(R_t g_n))(x) \\ &= (R_s(R_t(R_n(g))))(x) = (R_s(R_t(g^{-1}V_n g))(x) \\ &= (R_s(g(V_n g)^{-1}\cdot(V_t g)^{-1}\cdot V_{t+n}g))(x) \\ &= (R_s((g^{-1}V_n g)^{-1}\cdot(g^{-1}V_t g)^{-1}\cdot g^{-1}V_{t+n}g))(x) \\ &= (R_s((R_n g)^{-1}\cdot(R_t g)^{-1}\cdot R_{t+n}g))(x) \\ &= ([R_s(R_n g)]^{-1}\cdot[R_s(R_t g)]^{-1}\cdot[R_s(R_{t+n}g)])(x) \\ &= ([R_n(R_s g)]^{-1}\cdot[R_t(R_s g)]^{-1}\ [R_{t+n}(R_s g)])(x) \\ &= ([\tilde{R}_n f]^{-1}\cdot[\tilde{R}_t f]^{-1}\ [\tilde{R}_{t+n} f])(s,x), \end{aligned}$$

and (3.5) is satisfied.

For use later we will want the following condition satisfied: If $\alpha(t)$ is the character of H_1 defined by $(\alpha(t),\lambda) = \lambda(t)$ [recall that H_1 is a subgroup of the character group of R_d] then there is an extension $\gamma(t)$ of $\alpha(t)$ to all of H such that

$$(\gamma(t+s),f) = (\gamma(t)\cdot\gamma(s),f)\cdot(\gamma(s),\tilde{R}_t f) \tag{3.6}$$

for each $f \in H$.

(3.7) We say that (X, T_t) has quasi-discrete spectrum if the quasi-eigenfunctions G separate points in X.

4. Abstract system of quasi-eigenvalues and the existence problem

We begin by defining an abstract system of quasi-eigenvalues. Our goal will then be to show that for each such system there is a dynamical system (X, T_t) with an equivalent system of quasi-eigenvalues. By an abstract system of quasi-eigen values we mean an increasing family of abstract abelian groups $A_1 \subset A_2 \cdots \subset A_n \subset \cdots$; $A = \bigcup_{i=1} A_i$ where A_1 is a subgroup of the character group of R_d. Along with this we have homomorphisms $\sigma_t : A \to A$ for each

$t \in R_d$ such that $\ker \sigma_t = A_1$ and $\sigma_t A_{n+1} \subset A_n$, $n = 1, 2 \cdots$. Moreover we have

$$\sigma_t(\sigma_\mu(a)) = [\sigma_t(a)]^{-1} [\sigma_\mu(a)]^{-1} \sigma_{t+\mu}(a). \tag{4.1}$$

Further we let $\alpha(t) \in \hat{A}_1$ be the character for which $(\alpha(t), a) = (a, t)$; $t \in R_d$ and $a \in A$. We suppose that there is an extension $\gamma(t)$ of $\alpha(t)$ to all of A such that

$$(\gamma(t+s), a) = (\gamma(t) \cdot \gamma(s), a)(\gamma(s), \sigma_t a). \tag{4.2}$$

If (X, T_t) is a dynamical system with quasi-discrete spectrum $(H_n, \tilde{R}_t)$ then we say that $(H_n, \tilde{R}_t)$ is equivalent to the abstract system of quasi-eigenvalues $(A_n, \sigma_t, \gamma(t))$ if there is an isomorphism $\psi : H \to A$ such that $\psi = \text{id}$ on H_1 and $\psi : H_n \to A_n$ is an isomorphism, and $\psi \tilde{R}_t = \sigma_t \psi$.

THEOREM 4.1. *If $(A_n, \sigma_t, \gamma(t))$ is an abstract system of quasi-eigenvalues then there is a dynamical system (X, T_t) whose system of quasi-eigenvalues $(H_n, \tilde{R}_t)$ is equivalent to $(A_n, \sigma_t, \gamma(t))$.*

PROOF. Let $A = \bigcup_{n=1}^\infty A_n$ and $\theta_n : A_n \to A_{n+1}$ be the inclusion map. Let $X = \hat{A}$; then $(X, \hat{\theta}_n)$ forms an inverse system of groups and we let $X = \lim_n X_n$. We see that $X = \hat{A}$. We let $S_t : X \to X$ be defined by $S_t(x) = x \cdot \hat{\sigma}_t(x)$ where $\hat{\sigma}_t$ is the dual of σ_t. Since σ_t satisfies (4.1) we see easily that $\hat{\sigma}_t$ also does. We compute

$$\begin{aligned} S_{t+\mu}(x) &= x \cdot \hat{\sigma}_{t+\mu}(x) = x \cdot \hat{\sigma}_t(\hat{\sigma}_\mu(x)) \cdot \hat{\sigma}_t(x) \cdot \hat{\sigma}_\mu(x) \\ &= [x \cdot \hat{\sigma}_\mu(x)][\hat{\sigma}_t(x \cdot \hat{\sigma}_\mu(x))] \\ &= S_\mu(x) \cdot \hat{\sigma}_t(S_\mu(x)) = S_t(S_\mu(x)) \end{aligned}$$

and thus

$$S_{t+\mu} = S_t \cdot S_\mu. \tag{4.3}$$

We have that $\gamma(t)$ satisfies the following

$$(\gamma(t+s), a) = (\gamma(t)\gamma(s), a)(\gamma(s), \sigma_t(a)).$$

Thus we obtain

$$\gamma(t+s) = \gamma(t) \cdot \gamma(s) \cdot \hat{\sigma}_t(\gamma(s)) = \gamma(t) S_t(\gamma(s)). \tag{4.4}$$

We let $T_t(x) = \gamma(t) \cdot S_t(x)$ and observe

$$\begin{aligned} T_{t+s}(x) &= \gamma(t+s) \cdot S_{t+s}(x) \\ &= \gamma(t) S_t(\gamma(s)) \cdot S_t(S_s(x)) \\ &= \gamma(t) \cdot S_t(\gamma(s) S_s(x)) \\ &= T_t(\gamma(s) S_s(x)) = T_t(T_s(x)). \end{aligned}$$

Thus

$$(4.5) \qquad T_{t+s} = T_t \cdot T_s.$$

We wish to observe that S_t is an automorphism of X. We observe that $S_0 = \text{id}$ and $S_0 = S_{t-t} = S_t \cdot S_{-t} = S_{-t} \cdot S_t$; therefore S_t is one-to-one and onto.

We let $G'_n = \{\lambda a : a \in A_n \text{ and } \lambda \in C\}$. Hence we look at a as a function on X. It is immediate that $G'_n \subset G_n$. Not so immediate is the needed fact that $G'_n = G_n$. We want two things which we will attempt to prove in the next section.

1) Sufficient conditions on A such that (X, T_t) is ergodic.

2) The fact that in an ergodic system quasi-eigenfunctions of distinct classes are orthogonal. And for two elements $f, g \in G_n$ either $f \perp g$ or $f = kg, k$ constant.

With those two results we easily get $G'_n = G_n$. We see this as follows: Each $G'_n \subset G_n$ and $G'_{n+1} - G'_n \subset G_{n+1} - G_n$. Thus suppose $f \in G_{n+1} - G_n$. Under these conditions $f \perp G_{m+1} - G_m \supset G'_{m+1} - G'_m$ for $m > n$. Thus $f \perp G' - G'_n$. Since the characters span $L^2(X)$ we see that $f \in \text{span } G'_n$. Thus span $G'_n =$ span G_n. If $G'_n \neq G_n$ then there would exist $f \in G_n$ such that $f \perp G'_n$ so span $G'_n \neq$ span G_n. So we conclude $G'_n = G_n$.

We see that H_n may now be defined $H_n = \{(a, \gamma(t)) \cdot \sigma_t a : a \in A_n\}$. In the system (X, T_t) we now wish to compute what $\tilde{R}_t$ is on $H = \bigcup_{n=1}^{\infty} H_n$. Let $(a, \gamma(s)) \cdot \sigma_s a$ $(a \in A_n)$ be in H_n. To compute $\tilde{R}_t$ we must do the following: for each fixed a we apply V_t and compute the quasi-eigenfunction.

Thus

$$\begin{aligned}(V_t(a, \gamma(s)) \cdot \sigma_s a)x &= ((a, \gamma(s)) \cdot \sigma_s(a), \gamma(t) S_t(x)) \\ &= (a, \gamma(s))(\sigma_s a, \gamma(t))(\sigma_s(a), x\hat{\sigma}_t(x)) \\ &= (\sigma_s a, \gamma(t))(\sigma_t \sigma_s(a), x)(a, \gamma(s))(\sigma_s(a), x).\end{aligned}$$

Thus

$$(4.6) \qquad \tilde{R}_t[(a, \gamma(s)) \cdot \sigma_s a] = (\sigma_s a, \gamma(t)) \cdot \sigma_t \sigma_s a.$$

We wish to show that the systems $\{H_n, \tilde{R}_t\}$ and $\{A_n, \sigma_t\}$ are equivalent. If $a \in A$ let

$$(4.7) \qquad \psi(a) = (a, \gamma(s)) \sigma_s a \in H.$$

The mapping ψ is a homomorphism and it is onto. In order to see that it is one-to-one we need to show $(a, \gamma(s))(\sigma_s a, x) = 1$ for all s and x implies a is the trivial character. Fix s; then this means that $(a, \gamma(s))\sigma_s a \equiv 1$. $|(a, \gamma(s))| = 1$ thus $\sigma_s(a)$ is a constant character so $\sigma_s a = 1$. This is true for arbitrary s. The kernel $\sigma_s = A_1$ so

$a \in A_1$. We now have $(a, \gamma(s)) = 1$ for all s and $a \in A_1$. The ergodicity of (X, T_t) thus implies $a \equiv 1$.

We must now only check that $\psi\sigma_t = \tilde{R}_t\psi$. In order to do this we must give another computation for $\tilde{R}_t$. We observe

$$\tilde{R}_t(a, \gamma(s))\sigma_s a = R_t \cdot R_s(a),$$

but we have $R_t \cdot R_s = R_s \cdot R_t$ so that

$$\begin{aligned} \tilde{R}_t(a, \gamma(s))\sigma_s a &= R_s R_t(a) = R_s(a, \gamma(t))\sigma_t(a) \\ &= (\sigma_t(a), \gamma(s))\, \sigma_s \sigma_t(a). \end{aligned} \tag{4.8}$$

We use (4.8) in the following computation

$$\psi(\sigma_t a) = (\sigma_t(a), \gamma(s))\sigma_s\sigma_t a$$

$$\tilde{R}_t(\psi a) = \tilde{R}_t[(a, \gamma(s))\sigma_s a] = (\sigma_t(a), \gamma(s))\sigma_s\sigma_t(a)$$

thus $\psi\sigma_t = \tilde{R}_t\psi$ and the theorem is complete.

We wish now to examine the system (X, T_t) defined in the previous theorem and see whether it is ergodic.

THEOREM 4.2. *The system (X, T_t) defined in the previous theorem is ergodic.*

PROOF. (Omitted. Numerous misprints in the manuscript made the proof obscure. Nevertheless, the statement is true.)

5. Orthogonality of quasi-eigenfunctions

In this section we wish to examine the quasi-eigenfunctions and to establish some of their properties.

(5.1) If f and $g \in G$ and if $R_t f = R_t g$ for all $t \in R_d$ then f is a constant multiple of g, moreover the constant has modulus 1. We see this since

$$V_t \frac{f}{g} = \frac{R_t f \cdot f}{R_t g \cdot g} = \frac{f}{g} \quad .$$

Thus by ergodicity f/g = constant.

(5.2) Let $P : G \to H$ be defined by $Pf(x) = (R_s f)(x)$. That is, each element of G is mapped to its quasi-eigen value. By (5.1) we see that the kernel of P is K, the multiplicative group of complex numbers of modulus 1. Thus we may split G. That is, $G = K \times \Phi$. We let each $G_n = K \times \Phi_n$. We see that P restricted to Φ_n is an isomorphism into H_n.

For the next lemma we use the following notation: If $s_1, s_2 \cdots s_n$ are n real numbers let $\tilde{P}(s_1 \cdots s_n) = \tilde{R}_{s_1} \circ \tilde{R}_{s_2} \circ \cdots \circ \tilde{R}_s$.

LEMMA 5.3 $H_n = \bigcap_{s_i \in R_d} \ker \tilde{P}(s_1, \cdots, s_n) \equiv \Sigma_n$.

PROOF. We prove this by induction. Let $n = 1$. If $f \in H_1$ then $f : R_d \to \mathbb{C}$ and $\tilde{R}_s f = 1$ so $f \in \Sigma_1$. Then $H_1 \subset \Sigma_1$. Suppose $f \in \Sigma_1$ then $\tilde{R}_t f = 1$ for all $t \in R_d$. Since V_t is ergodic we see that f is a function of s alone, that is $f : R_d \to C$. Since $f \in H$ also we have $f = R_s g$ for some $g \in G$. Now f is a function of s alone so $g \in G_1$ and $f \in H_1$.

Suppose our statement is true for $n = p$ that is $H_p = \Sigma_p$. We wish to show that this implies $H_{p+1} = \Sigma_{p+1}$. Let $f \in H_{p+1}$ then for any s we have $\tilde{R}_s f \in H_p = \Sigma_p$. Now if $\tilde{R}_s f \in \Sigma_p$ we have immediately that $f \in \Sigma_{p+1}$ so $H_{p+1} \subset \Sigma_{p+1}$.

Now let $f \in \Sigma_{p+1}$. Then for any t we have $\tilde{R}_t f \in \Sigma_p = H_p$ so $\tilde{R}_t f \in H_p$ for all t. Since $f \in H$ we see that $f = R_s g$ for $g \in G$ so $R_t R_s g \in H_p$ for each fixed t. We must show $g \in G_{p+1}$. We can show this by showing that for each fixed s the function $g_s = R_s g \in G_p$. Since $R_t g_s \in H_p$ for each t we see that $g_s \in G_p$ which completes the argument showing $\Sigma_{p+1} \subset H_{p+1}$. Thus $\Sigma_{p+1} = H_{p+1}$ and the induction is complete.

LEMMA 5.4. *If H_1 has no elements of finite order then neither does Φ_{n+1}/Φ_n.*

PROOF. Consider $P : \Phi_{n+1} \to H_{n+1}$. This is an isomorphism. For each $\tilde{P}(s_1 \cdots s_n) \in \Sigma_n$ we see $\tilde{P}(s_1 \cdots s_n) \circ P : \Phi_{n+1} \to H_1$. We must show that if $f \notin \Phi_n$ then neither does any power of f. This is the same as showing that if $Pf \notin H_n$ then neither is any power of Pf. If $Pf \notin H_n$ then by the previous lemma there is a $\tilde{P}(s_1 \cdots s_n)$ such that $\tilde{P}(s_1 \cdots s_n)(Pf) \neq 1$. Since H_1 is assumed to have no elements of finite order we see that no power of Pf is in kernel $\tilde{P}(s_1 \cdots s_n)$ and then no power of $Pf \in H_n$ so Φ_{n+1}/Φ_n has no elements of finite order.

By $\mathrm{sp}(G_n)$ we mean the closed linear span of G_n in $L^2(X, \mu)$.

LEMMA 5.5. *If $g \in G_{n+1}[\Phi_{n+1}]$ and $g \notin \mathrm{sp}(G_n)[\Phi_n]$ then $g \perp h$ for each $h \in G_n[\Phi_n]$.*

PROOF. Let H be the orthocomplement of $\mathrm{sp}(G_n)$ in $\mathrm{p}(G_{n+1})$, i.e., $H = \mathrm{sp}(G_{n+1}) \ominus \mathrm{sp}(G_n)$. Let $g = g_1 + g_2$ where $g_1 \in \mathrm{sp}(G_n)$ and $g_2 \in H$. We must show that $g_1 = 0$.

We first observe $V_t : \mathrm{sp}(G_n) \to \mathrm{sp}(G_n)$ so since V_t is unitary $V_t H \subset H$. This is true for each t. Since $g \in G_{n+1}$ we see that $V_t g = f_t g$ where $f_t \in G_n$ for each fixed t. We have the following equations:

$$V_t g = V_t g_1 + V_t g_2$$

$$V_t g = f_t g_1 + f_t g_2$$

for each $t, f_t \in G_n$, $g_1 \in \mathrm{sp}(G_n)$ so $f_t g_1 \in \mathrm{sp}(G_n)$. We examine $f_t g_2$ and want to show that this is in H. We need only show for each $h \in G_n$ that $(h, f_t g_2) = 0$. We have $(h, f_t g_2) = (\bar{f}_t h, g_2) = 0$ since $\bar{f}_t h \in G_n$ and $g_2 \in H$. We then conclude $V_t g_1 = f_t g_1$ and $V_t g_2 = f_t g_2$. Since $|f_t| = 1$ we see that $|g_1|$ and $|g_2|$ are constant. If both g_1 and g_2 were different from zero we would conclude that $g_1/g_2 = k$ and thus $g_1 = kg_2$. This is impossible since $g_1 \perp kg_2$. Thus either g_1 or $g_2 = 0$; since $g \notin \mathrm{sp}(G_n)$ we get that $g_1 = 0$ and this concludes the proof.

THEOREM 5.6. *If H_1 has no elements of finite order then quasi-eigenfunctions corresponding to distinct quasi-eigenvalues are orthogonal.*

PROOF. It suffices to show that the elements of Φ are orthogonal. Since $\Phi = \bigcup_{i=1}^{\infty} \Phi_i$ we proceed by induction. We must observe that the elements of Φ_1 are orthogonal. Let f and $g \in \Phi_1$ then $V_t f = \lambda(t) f$, $V_t g = \Theta(t) g$, $\Theta(t) \neq \lambda(t)$. Since V unitary $(f, g) = (V_t f, V_t g) = \lambda(t)\,\overline{\theta(t)}(f, g)$. Thus either $\Theta(t) = \lambda(t)$ or $(f, g) = 0$. The first case is ruled out so $(f, g) = 0$.

We now assume that the elements of Φ_n are mutually orthogonal and we must show that this is true of Φ_{n+1}. Part of what we must show is that if $g \in \Phi_{n+1} - \Phi_n$ then $g \perp \Phi_n$. The previous lemmas show that if $g \notin \mathrm{sp}(\Phi_n)$ then $g \perp \Phi_n$. We must rule out the possibility that $g \in \mathrm{sp}(\Phi_n)$. We suppose that this is true and arrive at an absurdity. If $g \in \mathrm{sp}(\Phi_n)$ then $g = \sum_\alpha c_\alpha g_\alpha$ where $g_\alpha \in \Phi_n$ and $c_\alpha = (g, g_\alpha)$. We decompose Φ_n into cosets by Φ_{n-1}. We index these cosets A_p. In each coset we index the functions $g_{\beta\delta}$.

Thus we have

$$g = \sum_\beta \sum_\delta c_{\beta\delta} g_{\beta\delta};$$

since $g \in \Phi_{n+1} - \Phi_n$ we have

$$* \qquad V_t g = f_t g = \sum_\beta \sum_\delta c_{\beta\delta} f_t g_{\beta\delta}.$$

We observe $f_t \in G_n - G_{n-1}$ and then $f_t = k_t g^t_{\beta_0\delta_0}$, k_t a constant depending only on t, $|k_t| = 1$ and $g^t_{\beta_0\delta_0} \in \Phi_n - \Phi_{n-1}$ so $g_{\beta_0\delta_0} \neq 1$. Thus we obtain

$$** \qquad V_t g = k_t \sum_\beta \sum_\delta c_{\beta\delta} g^t_{\beta_0\delta_0} g_{\beta\delta}.$$

We also have

$$*** \qquad V_t g = \sum_\beta \sum_\delta c_{\beta\delta} V_t g_{\beta\delta}$$

$$= \sum_\beta \sum_\delta c_{\beta\delta} f^t_{\beta\delta} g_{\beta\delta}$$

where $f^t_{\beta\delta} \in G_{n-1}$. Thus we may write $f^t_{\beta\delta} = P^t_{\beta\delta} g^t_{\beta'\delta'}$ where $P^t_{\beta\delta}$ is a constant of modulus 1 and $g^t_{\beta'\delta'} \in \Phi_{n-1}$.

Then

**** $$V_t g = \Sigma_\beta \Sigma_\delta c_{\beta\delta} P^t_{\beta\delta} g_{\beta'\delta'} g_{\beta\delta}.$$

Putting ** and **** together we can write

$$V_t g = \Sigma_\beta \Sigma_\delta d^t_{\beta\delta} g^t_{\beta_0\delta_0} g_{\beta\delta} g^t_{\beta_0\delta_0} \in \Phi_n - \Phi_{n-1}$$
$$\left|d^t_{\beta\delta}\right| = c_{\beta\delta}$$

and

$$V_t g = \Sigma_\beta \Sigma_\delta e^t_{\beta\delta} g_{\beta\delta} \qquad \left|e^t_{\beta\delta}\right| = c_{\beta\delta}.$$

If we let $g^t_{i'k'} g_{ik} = g_{\pi(i,k)}$ then we see by comparing coefficients that $\left|a_{\beta\delta}\right| = \left|a_{\pi(\beta,\delta)}\right| = \left|a_{\pi^2(\beta,\delta)}\right| \cdots$. Since Φ_n/Φ_{n-1} has no elements of finite order we see that the sequence $(g^t_{i'k'})^n g_{ik}$ has no repetitions. Then the coefficient sequence $\left|a_{\beta\delta}\right| = \left|a_{\pi(\beta,\delta)}\right| = \left|a_{\pi^2(\beta,\delta)}\right| = \cdots$ is infinite. This is only possible if $\left|a_{\beta\delta}\right| = 0$. Consequently g must be constant. But $g \in \Phi_{n+1} - \Phi_n$ so g is not constant. Thus we have shown $g \in \Phi_{n+1} - \Phi_n$ and then $g \perp \Phi_n$.

We now need only show that if g_1 and $g_2 \in \Phi_{n+1}$ and $g_1 \neq g_2$ then $(g_1, g_2) = 0$. From the previous argument and the induction hypothesis we may assume g_1 and $g_2 \in \Phi_{n+1} - \Phi_n$. If $(g_1, g_2) \neq 0$ then

$$(V_t g_1, V_t g_2) = (g_1, g_2) \neq 0.$$

Letting $V_t g_i = f^i_t g$ we get

$$0 \neq (f^1_t g_1, f^2_t g_2) = (f^1_t \bar{f}^2_t \bar{g}_1 g_2),$$

$f^1_t f^2_t \in G_n$ thus there is a $g \in \Phi_n$ and a constant k_t such that $k_t g = f^1_t \bar{f}^2_t$ and therefore

$$0 \neq (k_t g, \bar{g}_1 g_2) = k_t(g, \bar{g}_1 g_2).$$

Since $\bar{g}_1 g_2 \in \Phi_{n+1} - \Phi_n$ and $g \in \Phi_n$ we have $(g, \bar{g}_1 g_2) = 0$ from the previous arguments. This is the needed contradiction and concludes the theorem.

6. Uniqueness

The purpose of this section is to show that if (X, T_t) and (X', T'_t) are two systems with equivalent quasi-eigenvalues then the systems are isomorphic. To do this we first want to set some algebraic background.

Let us recall the various maps associated with (X, T_t). If $f \in H$ we let f^s be the element of G whose value at x is $f^s(x) = f(s,x)$. With this notation we recall all our fundamental mappings

$$R_t g = g^{-1}V_t(g) \text{ for each } t \in R.$$

$R_t : G \to G$ with $G_n \to G_{n-1}$. We have $R : G \to H$ which is given by

* $$(Rg)(t,x) = (R^t g)(x) \text{ or, said in another way,}$$

** $$(Rg)^t = R_t g.$$

Finally we have $\tilde{R}_t : H \to H$ which were defined as follows: if $f \in H$ then there is a $g \in G$ such that $f(s,x) = (R_s g)(x) = (R_g)^s(x)$. We let $(\tilde{R}_t f)(s,x) = (R_t(R_s g))(x)$. Using our notation we see that $f^s = R_s g$ thus

*** $$(\tilde{R}_t f)^s = R_t f^s.$$

For (X', T'_s) we define the same mappings, V'_t, R'_t, $\tilde{R}'_t, R'$. We say that the two systems $(H, \tilde{R}_t)$ and $(H', \tilde{R}'_t)$ are equivalent if there is an isomorphism $\psi : H \to H'$ for which $\psi\tilde{R}_t = \tilde{R}'_t\psi$. We recall that

$$\begin{array}{ccccccc}
1 \to K \to & K \times \Phi & \overset{R}{\underset{P}{\leftrightarrows}} & H & \to 1 \\
 & \Theta\big\downarrow & & \big\downarrow\psi & \\
1 \to K \to & K \times \Phi' & \overset{R'}{\underset{P'}{\leftrightarrows}} & H' & \to 1
\end{array}$$

where $K \times \Phi \approx G$, $K \times \Phi' \approx G'$, $P(H) = \Phi$, $P'(H') = \Phi'$, $PR = \text{id}$, $P'R' = \text{id}$. We define Θ as follows : let $g = k \cdot f$, $k \in K$, $f \in \Phi$, $\Theta(g) = \Theta(k \cdot f) = k \cdot \eta(f)$ where

$$\eta(f) = P'\psi R(f).$$

I wish to show that P' can always be chosen so that

(6.1) $$\Theta f^s = (\psi f)^s \text{ for each } f \in H \text{ and } s \in R.$$

Recall $f^s \in G$ as the above makes sense. If $g \in G$ let $g = k(g) \cdot \phi(g)$ be its decomposition and similarly if $g' \in G', g' = k'(g') \cdot \phi'(g')$. To show (6.1) we must show how first

$$\Theta(f^s) = \Theta(k(f^s)\phi(f^s)) = k(f^s) \cdot \eta(\phi(f^s))$$

$$(\psi f)^s = k'((\psi f)^s) \cdot \phi'((\psi f)^s),$$

thus we must show

$$k(f^s) \cdot \eta(\phi(f^s)) = k'((\psi f)^s) \cdot \phi'((\psi f)^s).$$

I claim first that

$$\eta(\phi(f^s)) = \phi'((\psi f)^s). \tag{6.2}$$

In order to show this I show first

$$\psi(R(f^s)) = R'((\psi f)^s). \tag{6.3}$$

Consider

$$(R(f^s))(t,x) \overset{*}{=} (R_t(f^s))(x) \overset{***}{=} (\tilde{R}_t f)^s(x) = (\tilde{R}_t f)(s,x).$$

We now apply ψ to the function $(\tilde{R}_t f)(s,x)$ and we get $(\psi(\tilde{R}_t f))(s,x) = (\tilde{R}_t (\psi f))(s,x)$ $= (\tilde{R}_t'(\psi f))^s(x) \overset{***}{=} (R_t'(\psi f)^s)(x) \overset{*}{=} (R'(\psi f)^s)(t,x)$; thus $(\psi(R(f)^s)(t,x) = (R'(\psi f))(t,x)$ which proves (6.3).

We now use (6.3) to prove (6.2).

$$\begin{aligned}\eta(\phi(f^s)) &= P'\psi R\phi(f^s) = P'\psi RPR(f^s)\\ &= P'\psi R(f^s) = P'R'(\psi f)^s = \phi'(\psi f)^s.\end{aligned}$$

Thus (6.2) holds. We must consequently only prove (6.4) to get (6.1):

(6.4) P' can be chosen so $k(f^s) = k'((\psi f^s))$ for all s.

Suppose we consider two arbitrary splittings

$$\begin{array}{c} 1 \to K \to K \times \Phi \underset{P}{\overset{R}{\leftrightarrows}} H \to 1 \\ \downarrow \psi \\ 1 \to K \to K \times \Phi'' \underset{P''}{\overset{R'}{\leftrightarrows}} H' \to 1. \end{array}$$

We wish to modify the second splitting so (6.4) is satisfied. We can write

$$\begin{aligned}(\psi f)^s &= k''((\psi f)^s)\phi''((\phi f)^s)\\ &= k(f^s)[k'(f^s)]^{-1} k''((\psi f)^s) \cdot \phi''((\psi f)^s).\end{aligned}$$

If we find a homomorphism $\Theta : \Phi'' \to K$ such that $\Theta(\phi''((\psi f)^s)$ $= [k(f^s)]^{-1} k''((\psi f)^s)$ then $P'(h') = \Theta(\phi''(h')) \cdot P''(h')$ will give us a splitting satisfying (6.4).

7. Conclusion

It is at this point that I am stuck. I can't find such a Θ. Maybe without more hypotheses it isn't true. If (6.4) is true then (6.1) holds and we can conclude as follows. Observe first that if $g \in G$ and $g = c \cdot f$ then

$$\begin{aligned}R_t'(\Theta g) &= R_t'(\eta f) = R_t'(P'\psi R(f)) = (R'P'\psi Rf)^t\\ &= (\psi Rf)^t = (\psi Rg)^t\end{aligned}$$

thus $(\psi Rg)^t = R_t'(\Theta g)$ for all t.

Observe next $\Theta(V_t g) = \Theta(g) \cdot \Theta(R_t g) = \Theta(g)\,\Theta(Rg)^t = \Theta(g) \cdot (\psi Rg)^t = \Theta(g)\,R_t'(\Theta g) = V_t'\Theta(g)$. Now we could conclude as in our other paper.

Concerning the other things: Section 5 proves (I think) the orthogonality of the distinct quasi-eigenfunctions. This does not use total minimality or complete ergodicity. Off hand it seems also to apply to the paper of Abramov and to Parry and Hahn. This would extend these results provided there are no mistakes.

F. Hahn
(Ex. Yale University
New Haven, Connecticut, U. S. A.)